A COMPARISON BETWEEN THE FLOW OF WATER AND MERCURY IN PIPES WITH A VIEW TO TESTING THE OSBORNE REYNOLDS' LAW OF SIMILARITY

BY

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WITH 3 PLATES AND 18 FIGURES IN THE TEXT

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INTRODUCTION

The law of similarity, as applied to the flow of fluids in pipes, was set forth by Osborne Reynolds in his fundamental paper of 1883 [1]. The law may, as shown below, be formulated in various ways but the essence of it is that the pressure-drop pr. unit-length in pipes, measured at a sufficiently large distance from the inlet, is determined to a certain degree by the dimensionless quantity $\frac{vd}{\nu}$, where v denotes the average velocity over the cross-section of the pipe, d the diameter of same and ν the dynamical viscosity equal to the ratio $\frac{\eta}{\varrho}$ of the coefficient of viscosity η and the density ϱ. The explanation of this law is to be sought in the fact that, when $\frac{vd}{\nu}$ has the same value in two cases, the flow is geometrically similar [2]. Reynolds' experiments were confined to water. Nevertheless, he was able to test to some degree the dependency of ν, the said quantity altering largely with the temperature. Reynolds' results made it most probable that the law of similarity would prove to hold good for all, or at any rate, for most fluids. During the last 10—15 years the question about the validity of the law for fluids, other than water, has, for reasons to be explained in the following, greatly gained in interest. Lord Rayleigh had pointed out that Osborne Reynolds' law for the flow through pipes is only a special application of a more general law relating to the resistance met by bodies moving through fluids in which the bodies are wholly submerged. The resistance is, for geometrically similar bodies, a function of the quantity $\frac{vd}{\nu}$, where d now denotes some or other linear dimension and of course the same in all cases. And just as with the flow in tubes, if $\frac{vd}{\nu}$ is the same for two similar bodies, the flow around these bodies is geometrically similar. This experience is of great value in the study of various technical systems such as propellers, aeroplanes etc., because

[1]) Phil. Trans. Roy. Soc. 1883.

[2]) Excellent representations of the subject considered have been given by R. v. Mises in his treatise: Elemente der technischen Hydromechanik I, 1914 and by A. H. Gibson in Chapters IV and V of The mechanical properties of fluids. 1923.

it allows us to carry out the investigations with models of smaller dimensions than those of the actual systems, the proper value of $\frac{vd}{\nu}$ being obtained either by an increase of v or a decrease of ν[1]. Now, ν is ab. 10 times less for water than for atmospheric air. If thus water is used in model-experiments with a system actually exposed to the action of an air-stream, the model may, with the same velocity of water and air, be made linearly ab. 10 times smaller than the actual system, or with larger models the velocity may be reduced. But, as a basis for such model-experiment, a test of the validity of the law of similarity, by a comparison of the resistance with water and air, is of course necessary. The verification was carried out by Stanton and Pannell who compared the resistance in tubes. The main results of their investigation are collected in a graph, which I have permitted myself to reproduce in plate I of this paper in order to make possible a direct comparison with my own results[2]. Stanton and Pannell's investigation very completely confirms the law of similarity. My own work is of the same kind as Stanton and Pannell's consisting of a test on the law of similarity by a comparison between the flow of water and mercury in the same pipes. Originally the sole purpose of the work was to establish a fairly exact basis for the design of the hydrodynamic circuit of the so-called jet-wave rectifier. Later on it was decided to carry out the investigation as a complete test on the law of similarity applied to mercury. Two considerations led me to this decision. In the first place it seemed not at all certain that the law of similarity would prove to hold good for mercury, a point already noted by Osborne Reynolds. As shown by Lord Rayleigh, the law of similarity may directly be derived by dimensional considerations, but only on the supposition that the flow is solely determined by the quantities v, ϱ, η and d. It is however *a priori* not precluded that the adhesion of the liquid to the wall of the tube or to the surface of the body, may not play a part in the phenomenon considered. And with regard to the said adhesion water and mercury are certainly extremely different, water as a rule wetting the wall and mercury ordinarily not. — If, however, the law of similarity should prove to hold good for mercury, that would mean that mercury could be used as a medium in model-experiments, with the effect that very small models might be used, ν for mercury being ab. 10 times smaller than for water, the ν of which is again, as stated above, ab. 10 times smaller than that of atmospheric air. Or, it would be possible to carry out hydrodynamical research at comparatively high values of $\frac{vd}{\nu}$, as will be seen from the experiments below. With regard to these experiments it should already in this place be remarked that they confirm the law of similarity. However, at the same time they reveal certain characteristics of the flow of mercury which undoubtedly must be attributed to the particular properties of this liquid.

[1]) With respect to the discussion of the validity of model-experiments the reader may be referred to the Dictionary of Applied Physics, Vol. V, pg. 191.

[2]) Phil. Trans. Roy. Soc. A. 214.

Formulations of The Osborne Reynolds Law of Similarity.

As stated above, the law of similarity may be formulated in various ways. If the pressure-drop per cm. is denoted by p, the diameter of the pipe by d, the average-velocity taken over the cross-section by v, the dynamical viscosity by ν and the density of the fluid by ϱ, the law may be written:

$$p = \varrho \frac{v^2}{d} \cdot F\left(\frac{vd}{\nu}\right).$$

If the head h corresponding to p and determined by $h = \frac{p}{\varrho g}$ is introduced, we get

1° $$h = \frac{v^2}{dg} \cdot F\left(\frac{vd}{\nu}\right).$$

The law thus expresses that $\frac{p}{\left(\varrho \frac{v^2}{d}\right)}$ or $\frac{h}{\left(\frac{v^2}{dg}\right)}$ is a function exclusively of the dimensionless quantity $\frac{vd}{\nu}$, which quantity is termed Reynolds' number.

To a second formulation one is led through the following consideration. If a fluid sweeps tangentially along the surface of a body which is entirely submerged in the fluid, it will act on the said surface with a tangential force which per cm^2 may be expressed by

2° $$R = \varrho v^2 f\left(\frac{vd}{\nu}\right).$$

This relation holds good for all geometrically similar systems, d indicating any linear dimension of the systems and of course the same in all. The expression may easily be derived by a simple dimensional consideration. It may be applied to pipes, or rather to parts of pipes at a considerable distance from the inlet, say at distances of 100 times the diameter. R, then, signifies the friction per cm^2 of the fluid against the wall. Between R and the pressure-drop per cm., p, the following relation must exist

3° $$R \cdot \pi d = p \frac{\pi}{4} d^2.$$

If R is introduced into 1° instead of p, Reynolds' law assumes the form:

4° $$R = \varrho v^2 \cdot \frac{1}{4} F\left(\frac{vd}{\nu}\right) = \varrho v^2 \cdot f\left(\frac{vd}{\nu}\right).$$

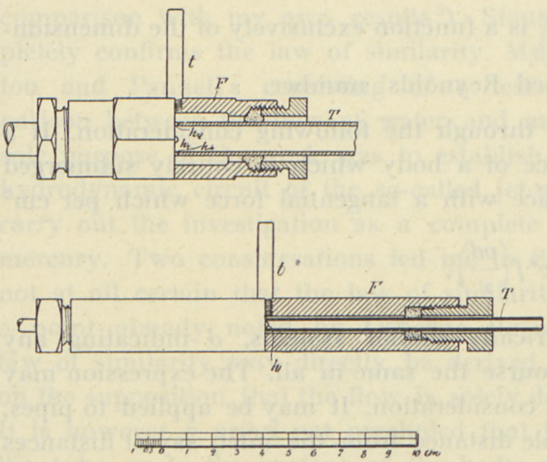
Finally we may introduce the new function $f\left(\frac{vd}{\nu}\right)$ into 1°, thereby getting

5° $$h = \frac{4 v^2}{dg} \cdot f\left(\frac{vd}{\nu}\right).$$

In the following we make use of the latter formula when determining the function $f\left(\frac{vd}{\nu}\right)$.

Research Apparatus.

The verification of the law of similarity was — apart from a series of preliminary experiments — carried out with two sets of tubes, one of glass and one of steel. The glas-tubes are in the following denoted G_1, G_2, G_3, G_4, G_5 and G_6, the steel-tubes S_1, S_3 and S_4. In table I below, the diameters of the tubes are entered in the second column. In the third the length l_1 of the inlet part i. e. the part of the tube preceding the research-part l, is introduced, l itself being entered in the fourth column of the table. The inlet-length was as a rule ab. 100 times the diameter of the tube, which length has proved sufficient for the creation of the final state of flow. The inlet-part of G_2 was abnormally short, i. e. only 50 times the diameter, and even this length seems sufficient, judged by the observations obtained with the said tube, the observations agreeing very well with those gained with the other tubes. The tube was originally supplied with an inlet-part of normal length and with two research-parts l_{01} and l_{12} of equal lengths. But the tube burst at the entrance of l_{01}, and l_{01} was thereupon used as inlet-part for l_{12}. The tube G_3 was also furnished with two research-parts and accordingly with three manometer-fittings with corresponding bores. It likewise burst at one of the bores, but was mended as described below. The bursting of the glass-tubes was a very frequent occurrence during the experiments and highly hampered the work. The reason for furnishing the tubes with more than two manometer-bores was mainly the desire of being able to check the homogeneousness of the caliber of the tubes by a comparison of the pressure-drops in the two parts of the tube.



Figs. 1 and 2.

The manometer-fittings, indicated above, are shown in fig. 1 and fig. 2, fig. 1 corresponding to the steel-, fig. 2 to the glass-tubes. In the glass-tubes a single hole h was bored at either end of the research-part. A steel-socket F with packing-boxes surrounded the part of the tube containing the hole, the packing material being soft rubber. A short tube t was screwed into the wall of the socket. Through this tube and through a rubber-tube the space between the research-tube and the socket was connected to one of the vertical glass-pipes constituting the pressure-measuring device. When the research tubes were made of glass, the sockets were rather long in order to diminish the strain in the tube-wall arising from the fastening of the socket. With the steel-tubes the sockets were shorter and instead of one single hole, four holes h_1, h_2, h_3, h_4 were bored in the wall of the tube, arranged of course in the same section. The interior edge of the each hole was always smoothed with a special scraper.

The final experiments were, with a single exception, carried out by means of the two arrangements indicated in fig. 3 to fig. 5. Fig. 3 and fig. 4 show the research apparatus used with mercury. The mercury flowed from a reservoir R_0 through the tube T_0 with the stop-cock V_0 to the reservoir R_1 at the top of the pressure-pipe T_1. In R_1 the mercury was kept at a constant level by means of an over-flow O. Previous to the experiment the flow from R_0 was regulated by the stop-cock V_0 in order to reduce the amount of mercury escaping through the over-flow. The aperture d of the pressure-pipe T_1 was $\frac{3}{4}$ of an inch. The available head was 1.67 m. The pressure-pipe was fitted at its lower end with a knee K_1 to which the end of the research-tube T was connected through a short piece of rubber-tube H_1. The research-tube proper was, as shown in fig. 4, fixed to a horizontal board S of wood the tube resting on wooden-blocks B. At the outlet-end it was connected, through a wide rubber-tube H_2, to a knee K_2 with a stop-cock V_1. The knee was furthermore fitted with a socket E into which nozzles N of variable aperture could be screwed. By means of such nozzles the velocity of the flow was adjusted. The manometer-device M, the lower part of which is indicated in fig. 4, consisted of two (or three) glass-tubes R_1 R_2 fixed on a vertical board together with a scale S. The scale was furnished with a slider G for the reading of the meniscus of the mercury-column. The tubes R_1 R_2 were connected with the two manometer-fittings F_1 F_2 through rubber-tubes H_3 H_4.

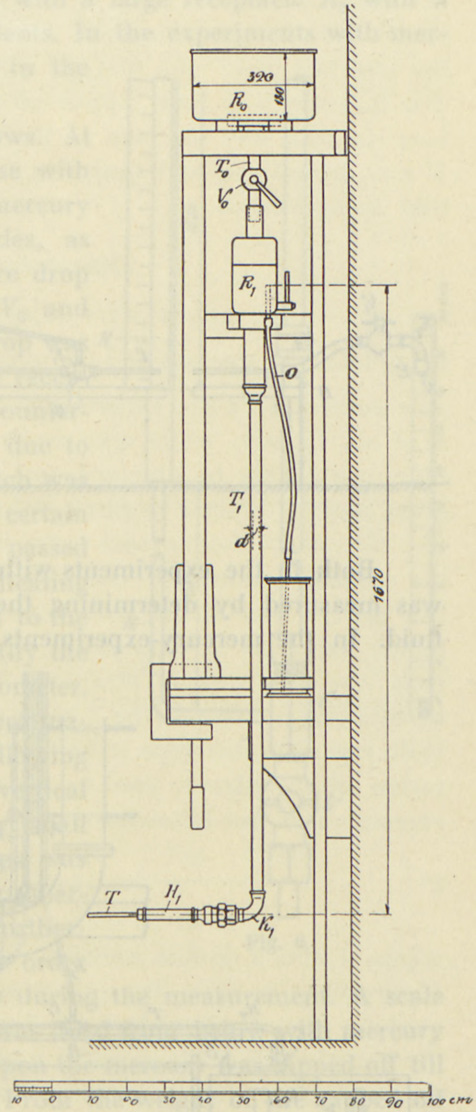


Fig. 3.

The apparatus used in the experiments with water is shown in fig. 5. The water was taken from a tank holding 37 m³. The available head was now 8 m. The research-tube was connected with a point of the pipe T' used for the filling of the tank. Between the pipe and the research tube a stop-cock V_0 was inserted. At the outlet-end of the tube a second stop-cock V_1 was applied. The manometer device M was of the same type as in the mercury-experiments.

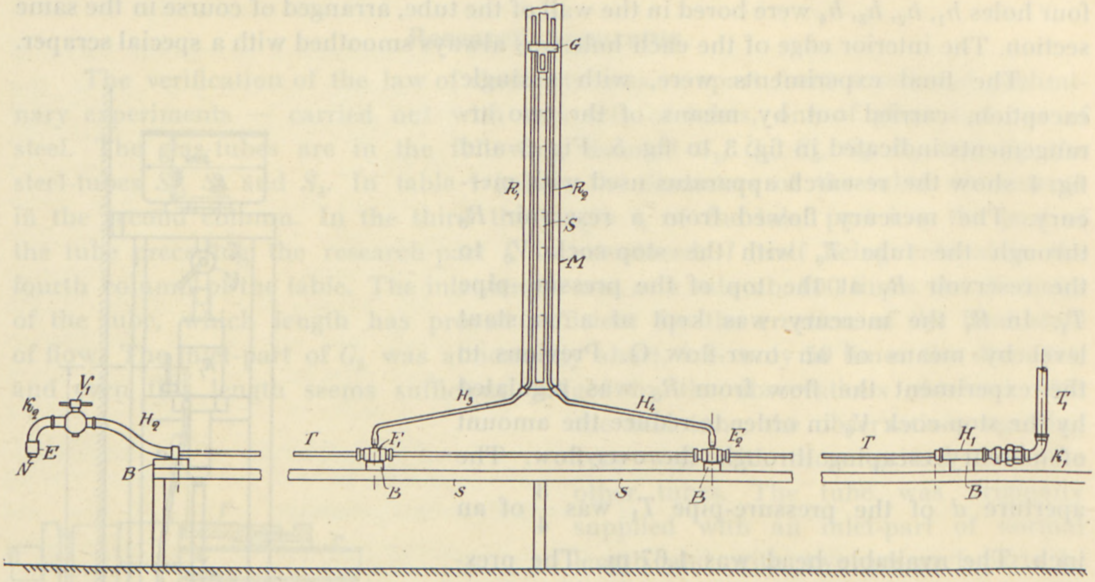


Fig. 4.

Both in the experiments with water and with mercury the velocity of the flow was measured by determining the time for the efflux of a certain weight of the fluid. In the mercury-experiments an ordinary decimal-balance was used on the

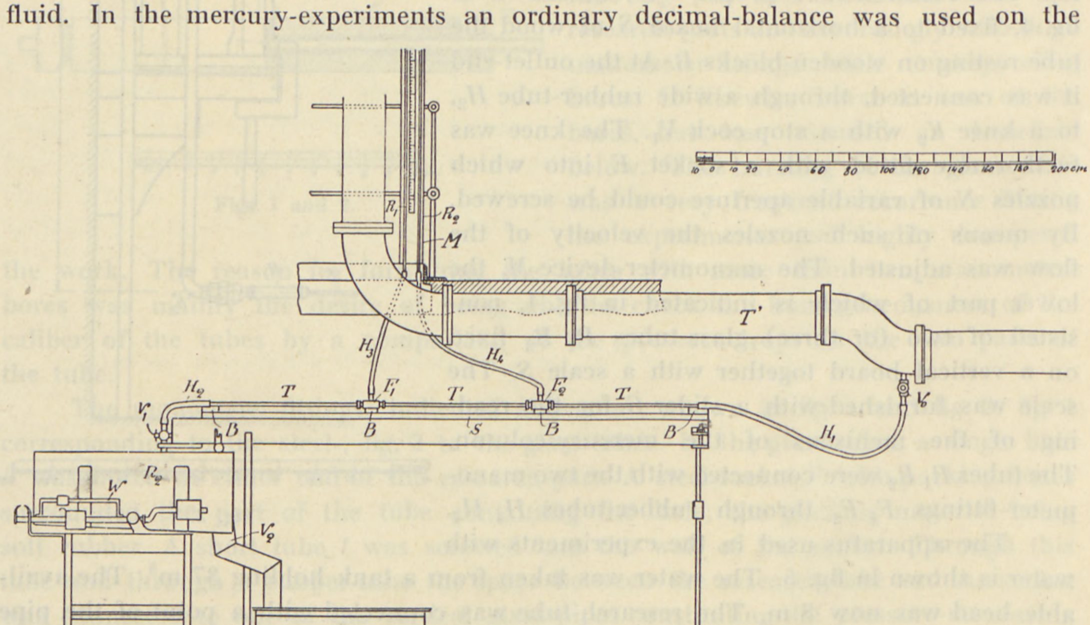


Fig. 5.

bridge of which the receptacle for the mercury was placed. The balance W, applied in the water-experiments, is pictured in fig. 5. It was constructed just for the kind of research-work considered and was furnished with a large receptacle R_2 with a bottom-valve V_2 for rapid exhaustion of the contents. In the experiments with mercury portions as large as 60 kg were weighed, in the water-experiments portions up to 100 kg.

The experiments were carried out as follows. At first the velocity of the flow, or, as was the case with water, the pressure-drop was adjusted. With mercury the velocity was regulated by means of nozzles, as indicated above. With water the desired pressure drop was obtained by means of the two stop-cocks V_0 and V_1 fig. 5. When the proper velocity or pressure-drop was produced the balance was adjusted so that the receptacle with its contents of fluid was more than counterbalanced by the weights. At the moment when, due to the flow, equilibrium was just reached, a stop-watch was started and the weights were increased by a certain amount. At the moment when the balance again passed zero the watch was stopped, its reading thus indicating the time for the efflux of a quantity of fluid equal to the added weight. During the efflux of the said quantity the pressure drop was repeatedly observed on the manometer.

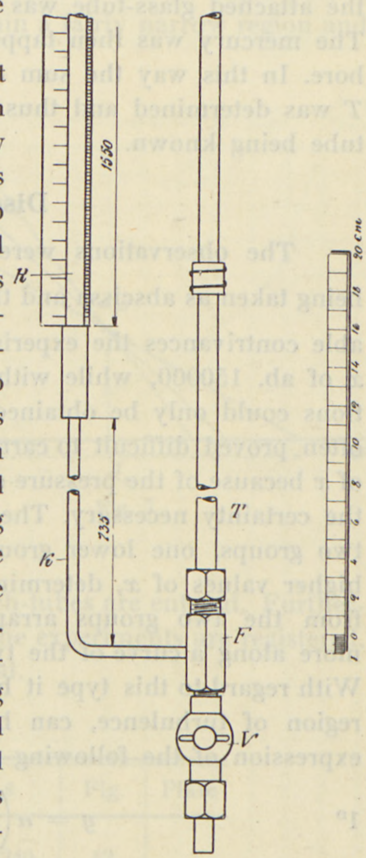


Fig. 6.

In the experiments with glass-pipes the determination of the diameter was carried out in the following simple manner. The pipe was adjusted in a vertical position and the bores in its wall were closed by small pieces of rubber-tube. The lower end of the pipe was connected to a stop-cock by means of a short rubber-tube. The said tube had an inlay of linen and was furthermore reinforced by a winding of copper-wire in order to prevent dilatation due to the interior pressure during the measurement. A scale was placed in the rear of the tube and the latter was filled from below with mercury up to the uppermost bore, or there about. Thereupon the mercury was tapped off till the meniscus was at level with the lowest bore. From the weight of the tapped off portion the average diameter for the research-part of the tube was calculated. With the same apparatus the tube might be calibrated, the mercury being tapped off in suitable small portions corresponding for instance to a lowering of the meniscus of 8—10 cm. In this way the calibrations of the tubes G_1 and G_3, mentioned below, were carried out.

In the determination of the diameter of the steel-tubes the apparatus shown in fig. 6 was used. The steel-tube T was screwed into a socket F fitted with a stop-cock V. Above the cock the socket was furnished with a steel-knee K lengthened by a calibrated glass-tube R. In the measurement of the diameter the bores in the steel-tube were closed in the same manner as in the case of the glass-tubes above and the apparatus was filled from below with mercury till the meniscus in the attached glass-tube was on a level with the uppermost bore in the steel-tube. The mercury was then tapped off till the meniscus was on a level with the lowest bore. In this way the sum of the cross-sections in the two connected tubes R and T was determined and thus also the cross-section of the steel-tube, that of the glass tube being known.

Discussion of the Observations.

The observations were represented graphically, the Reynolds number $x = \frac{vd}{\nu}$ being taken as abscissa and the quantity $y = \frac{R}{\varrho v^2} = \frac{h g d}{4 v^2}$ as ordinate. With the available contrivances the experiments with mercury could be extended up to values of x of ab. 150000, while with water, on account of the higher value of ν, observations could only be obtained for x-values up to ab. 50000. On the other hand it often proved difficult to carry the experiments with mercury down to smaller values of x because of the pressure-drops in this case being too small to be determined with the certainty necessary. The observations with a tube, therefore, as a rule fell into two groups, one lower group obtained with water, and another, corresponding to higher values of x, determined with mercury. It was examined whether the points from the two groups arranged themselves along one single curve and furthermore along a curve of the type to be expected from earlier experiments with water. With regard to this type it has proved, that results obtained with water, within the region of turbulence, can be represented with a high degree of exactness by an expression of the following form

$$1^\circ \qquad y = \alpha\left(1 - \frac{2000}{x}\right) + \beta\sqrt{\frac{1}{x}} \cdot \sqrt{1 - \frac{2000}{x} + \frac{8}{x}}$$

where α and β denote numeric constants which must be made to fit in with the observations[1]). Only the points originating from the experiments with water were used for the adaption, because the said points showed less uncertainty than those of the experiments with mercury and as a rule arranged themselves with a considerable exactness along a definite curve. The latter was drawn and a number of its points, as a rule 5, were thence used for the determination of α and β. Fig. 7 may illustrate in what manner the determination was carried out. Each of the five equations which were formed by introducing the selected points $x_1\, y_1$, $x_2\, y_2$ etc. into 1° was written in the form

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[1]) With respect to the deduction of this expression and the discussion of the various expressions which in the course of time have been suggested reference should be made to: R. v. Mises: Elemente der technischen Hydromechanik I 1914.

$$2^{\circ} \qquad \frac{\alpha}{A_1} + \frac{\beta}{B_1} = 1$$

and was thence represented graphically. The five straight lines were due to cut each other at a single point, the point with the coordinates α and β. As a rule it was found that their points of intersection gathered within a fairly narrow region and the coordinates of the middlepoint of the same were taken as smoothed-out values for α and β. The latter constants being thus determined a number of points of the smoothed-out curve, equally distributed over the entire range covered by the experiments with water and mercury, was calculated. The curves determined in this way are reproduced below. However, it should be noted that, after the first determination of α and β, small alterations in these constants were sometimes undertaken with a view to obtain a closer agreement with the points originating from the experiments with water.

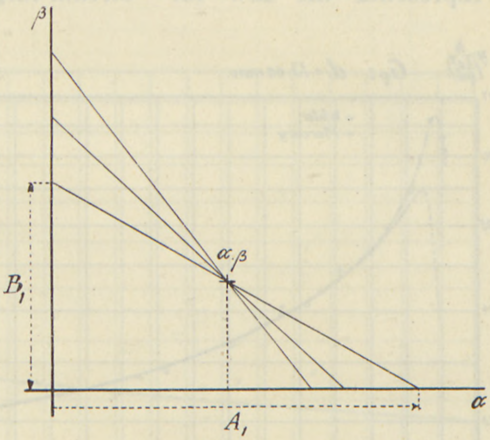


Fig. 7.

The Results.

In the following table the dimensions of the research-tubes are entered. Furthermore, the constants α and β of the curves representing the experiments are registered. Finally references to the figures or plates are introduced.

Table I.

Tube	d	l_1	l	α	β	Fig.	Plate
	cm	cm	cm				
G_1	1.538	15.0	100	0.00153	0.240	12	
G_2	1.360	65.5	55.35	0.00170	0.210	8	
G_3	0.8640	89.8	55.15 / 54.80	0.00180	0.205	..	II
G_4	0.5653	55.0	78.20	0.00195	0.210	9	
G_5	0.3372	35.0	95.37	0.00205	0.235	10	
G_6	0.1197	19.9	56.17	0.00239	0.220	11	
S_1......	1.505	150	100	0.00150	0.232	..	III
S_3......	1.037	100	100	0.00165 / 0.00250	0.210 / 0.200	13	
S_4......	0.756	75.0	100	0.00210	0.180	14	

It is seen from plate II that The Reynolds Law of Similarity holds good for mercury at any rate with tubes of larger apertures. Plate II represents the experiments with tube G_3. The ranges for water and mercury here overlap rather considerably, and the uncertainty of either set of points is nearly the same. A comparison with plate I shows that the curve of plate II coincides fairly closely with that of Stanton-Pannel's experiments. The G_3-experiment is carried near to the critical abscissa 2000. The curve-branch to the left of that abscissa represents the law for "stream-line" flow. The original observations and the

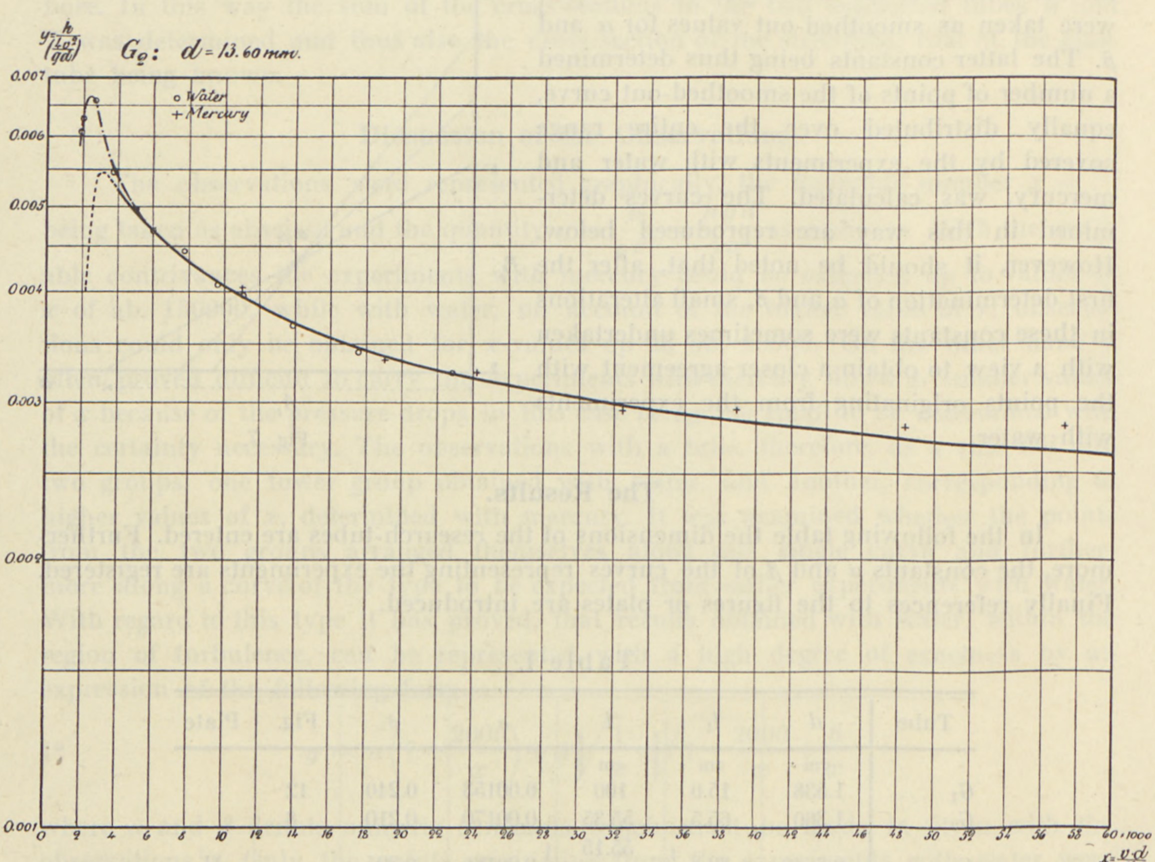


Fig. 8.

derived numerals have been collected in Appendix A below. The remainder of the experiments with glass-tubes may serve to illustrate various characteristic experiences. The G_2-experiment, fig. 8, has, with water, been carried down as far as to the critical abscissa 2000 and shows the sudden drop of the y-values. Upwards the experiment only extends to 60000. It illustrates a characteristic feature, which is rather often met with, consisting in the mercury-points deviating from the water-curve when a certain abscissa, viz. a certain velocity is reached. After the said abscissa the mercury-points are to be found over the water-curve. I hold the opinion that this is due to the mercury not wetting the wall of the tube. On account of this property the flow in the neighbourhood of the wall will, undoubtedly, largely depend on the adhesion to the wall, and said adhesion is certainly a poorly defined quality, unless particular and well-nigh impracticable precautions are taken. The want of definiteness of the flow with mercury is well known to me from numerous experiments with jet-holes of various shapes.

The anomaly reappears in the utmost part of the curve, fig. 9, representing

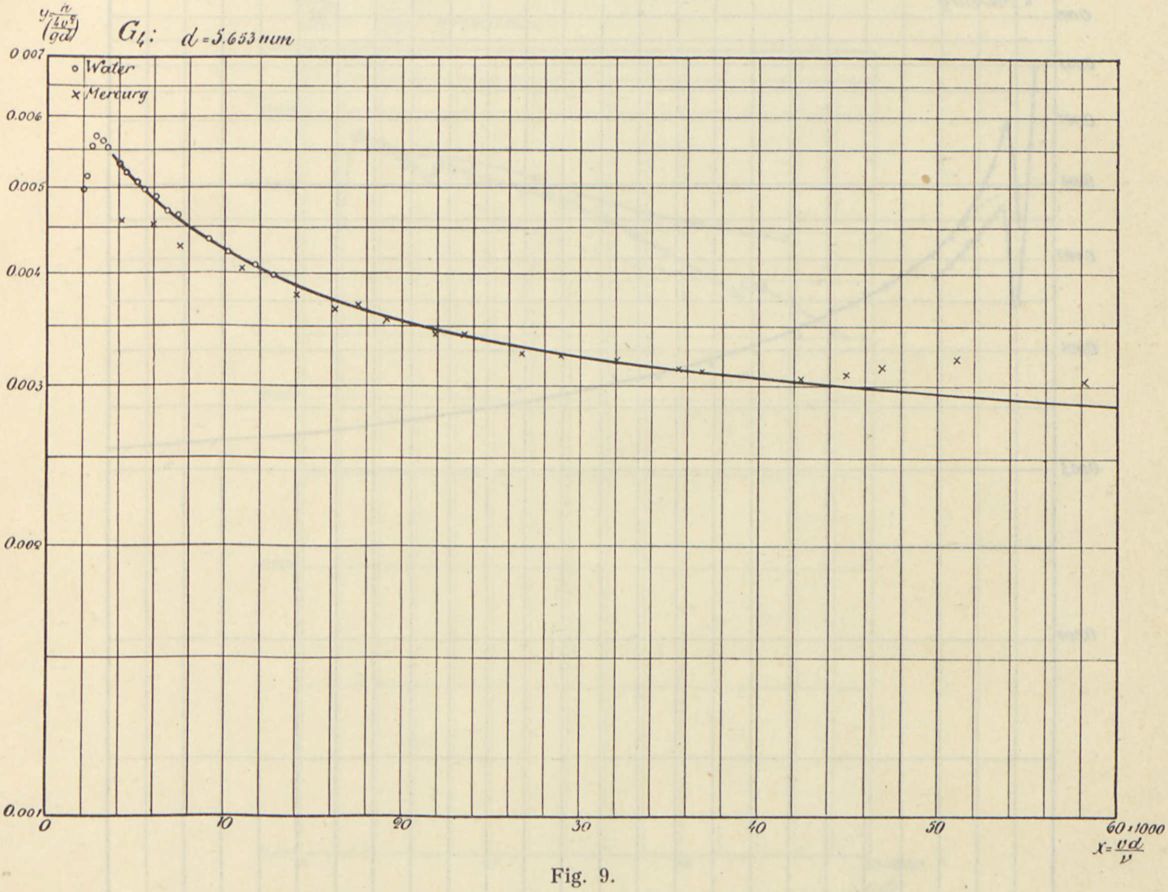


Fig. 9.

the G_4-experiments. The latter too, with water, extends downwards to the critical abscissa. A remarkable feature is that the mercury-points corresponding to small abscissae lie below the water-points. In the experiments with the narrow tube G_5, shown in fig. 10, the mercury-curve is decidedly higher than the water-curve, due undoubtedly to the wall-effect indicated above, the effect being of course more pronounced with a narrow tube than with one of greater diameter. The experiment shows that the sudden transition from the turbulent to the stream-line flow may be recorded in details[1]). Fig. 11 represents an experiment with the tube G_6. The

[1]) Compare L. Schiller: Zeitschrift für angewandte Mathematik und Mechanik I, 436. 1921.

experiment is only carried out with mercury and is to illustrate the anomalies characteristic to the flow of the said fluid in very narrow tubes, viz. both smooth alterations and sudden changes in the resistance. The dotted curve indicates the

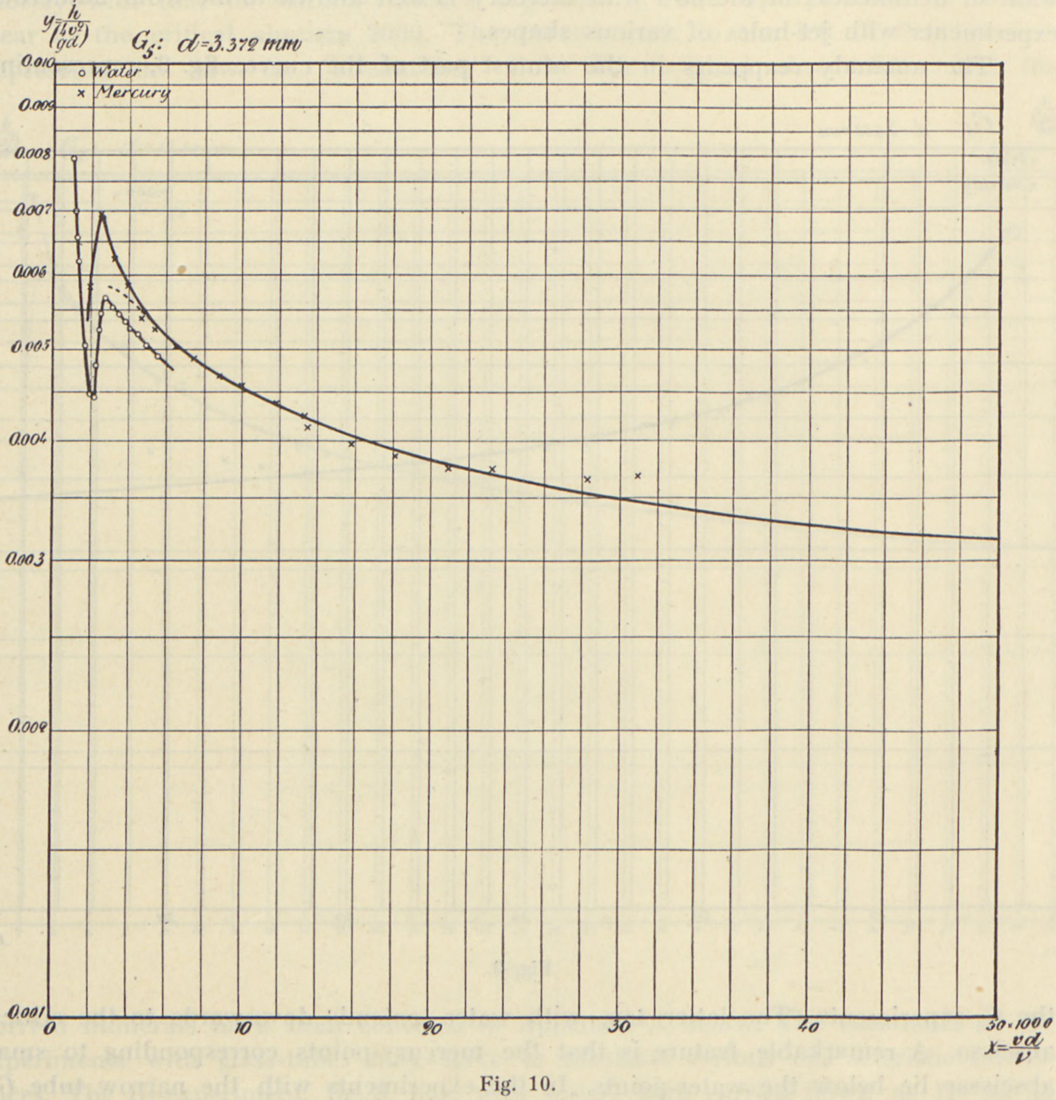


Fig. 10.

variation, which is to be expected with water. It represents an extrapolation from the experiments with the tubes G_1 to G_5 (comp. Appendix E). Near to the critical abscissa the mercury-points coincide fairly well with the water curve, but outside this range they as a rule lie higher. In Fig. 12, finally, the experiment with the wide tube G_1 is reproduced. The experiment was only carried out with water and serves to illustrate a source of error, which will be discussed in Appendix C.

We now proceed to consider the experiments with the steel-tubes. The law of similarity was fully confirmed through the experiment with the tube S_1, reproduced in plate III and with regard to its observations in appendix B. The experiment extends from the abscissa 10000 up to 150000 and exhibits a very small uncertainty with respect to the water-points. Less perfectly the law was verified in the

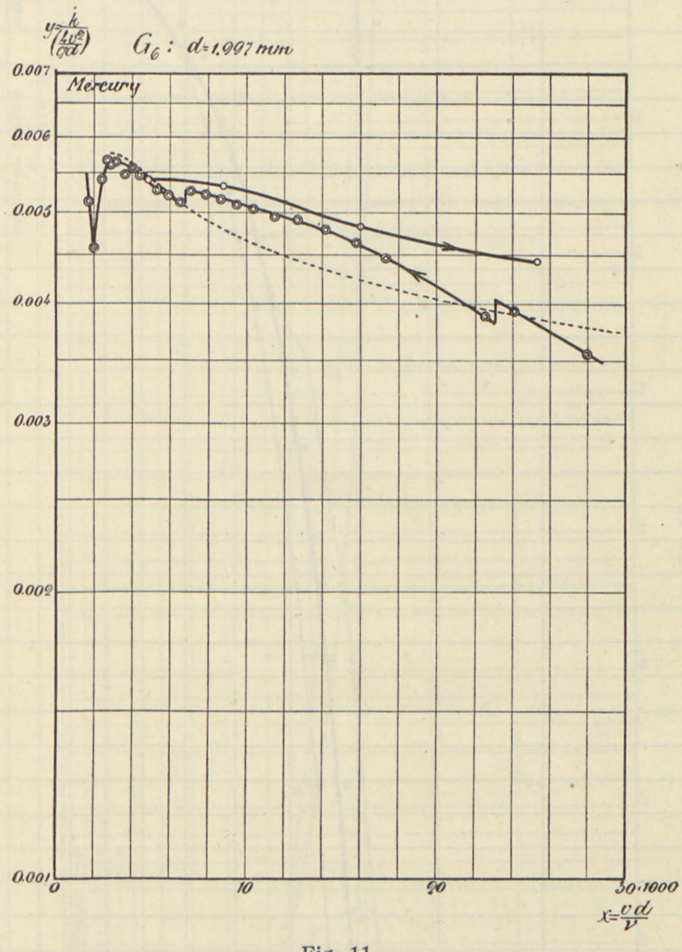


Fig. 11.

experiment with the tube S_3 fig. 13. The water-curve was here determined for flows in both directions. In a first experiment with mercury, points were found which, with greater values of $\frac{vd}{\nu}$ were situated considerably above the water-curve. The tube was then cleaned by drawing steel-wool through it, and the interior edges of the manometer-bores were scraped anew in order to remove any irregularities left. This resulted in the mercury-points being drawn nearer to the water curve without reaching the same. One more cleaning caused the mercury-points to coincide with the

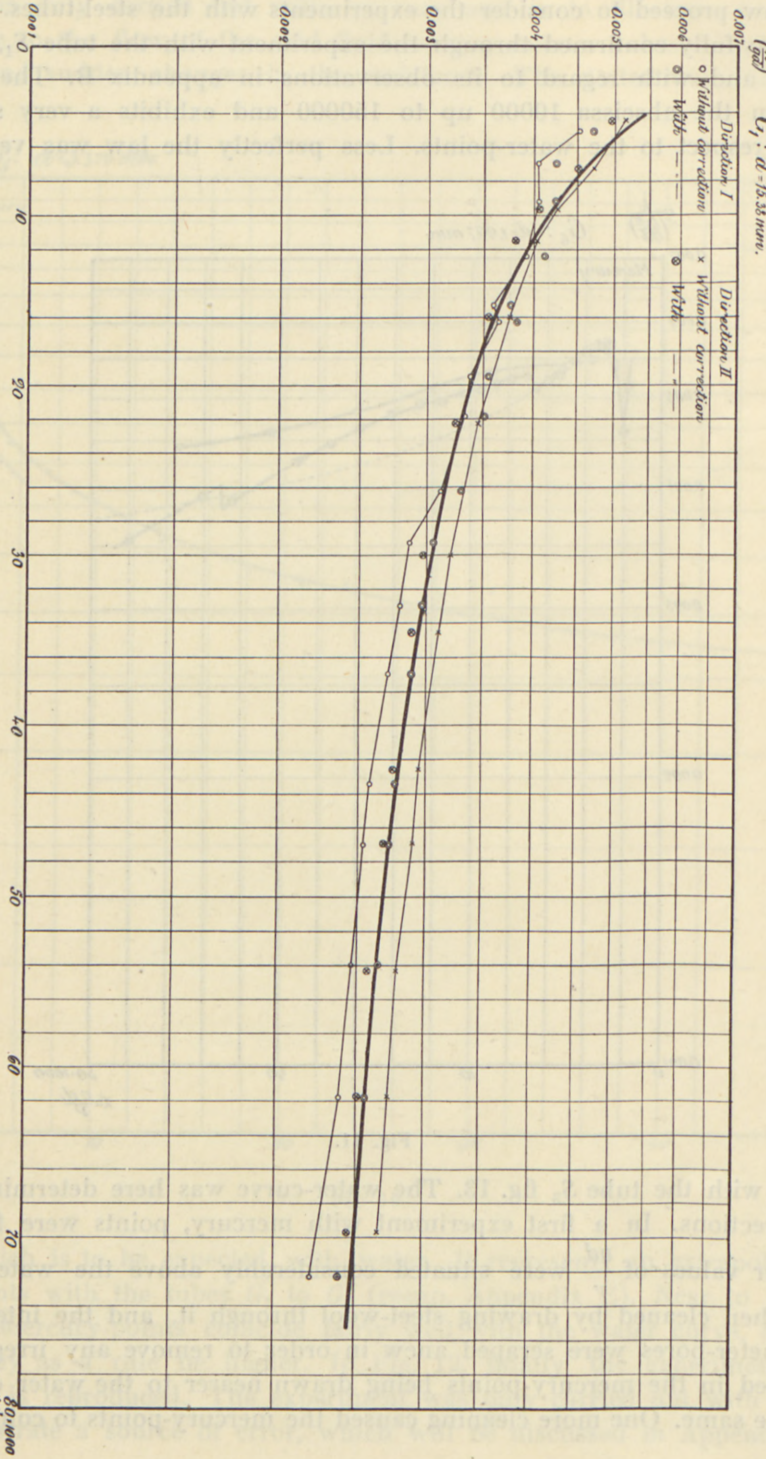
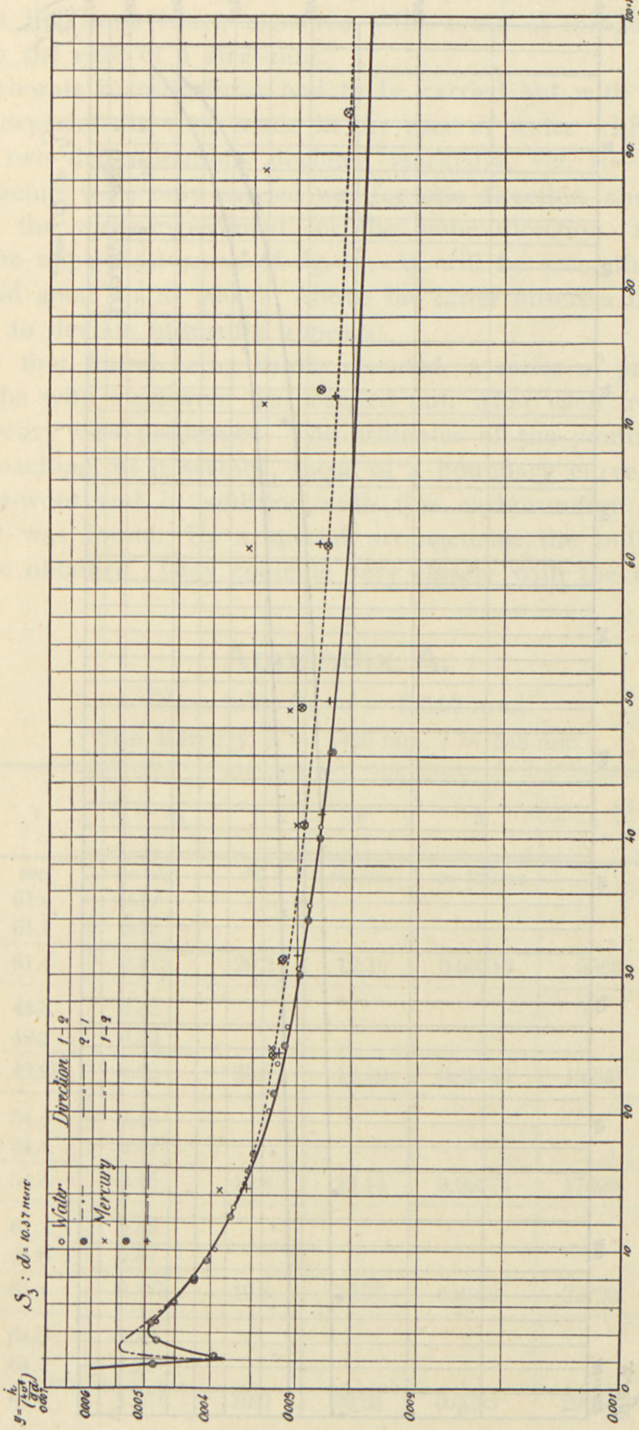

C_1 d=15.38 mm.
Direction I
o Without correction
● With — · —
Direction II
× Without correction
● With — · · —
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
10
20
30
40
50
60
70
80.000

Fig. 12.



S_3 : d = 10.37 mm
Water
Mercury
Direction 1-2
2-1
1-2
0
10
20
30
40
50
60
70
80
90

Fig. 13.

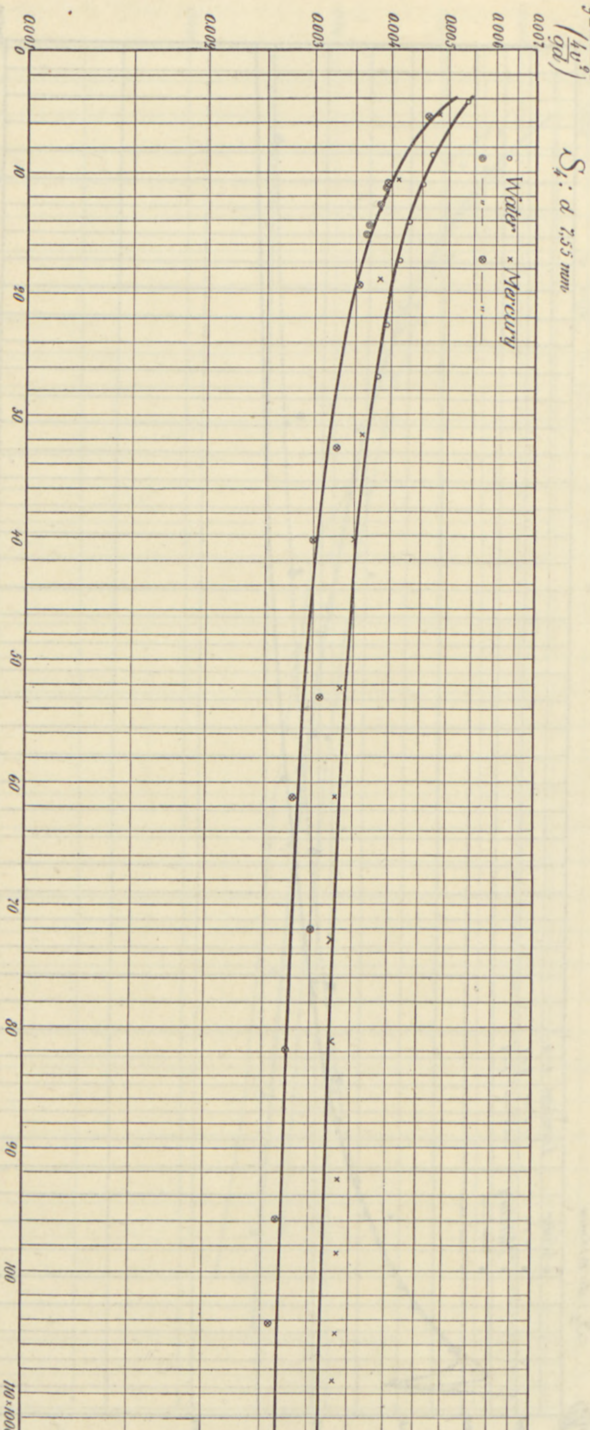

$y = \frac{h}{\left(\frac{4v^2}{gd}\right)}$
S_4: d 7.55 mm
Water
Mercury
0.000
0.002
0.003
0.004
0.005
0.006
0.007
0
10
20
30
40
50
60
70
80
90
100
110×1000
$x = \frac{vd}{\nu}$

Fig. 14.

water-curve up to $x = 50000$, but above this value they still deviated from the curve. Obviously it is the same effect, observed with some of the glass-tubes, which is now refound in the case of a steel-tube.

Rather elaborate investigations had to be carried out with the tube S_4. With this tube, too, experiments were made in the case of water with both directions of the flow. The two determinations deviated markedly, viz. 10—15 %. The experiments with mercury were only carried out for one direction and were accordingly compared with the water-experiment for the same direction. A first comparison was made in the uppermost curve of fig. 14. As will be seen, the law of similarity is found to hold good up to 90000. Above the latter abscissa the tendency of the mercury points to deviate upwardly appears.

The curve first found being rather elevated, a series of polishings of the interior of the tube with steel-wool was carried out. After each treatment an experiment with mercury was performed. The ordinates of the points observed steadily decreased, approaching, as it seemed, those of a boundary curve. After a last treatment with steel-wool and in addition with fine carborundum powder the lowest curve of fig. 14 was found. By a special arrangement the indicated observations with water were obtained. They coincide very closely with the mercury-curve.

Appendix A.

Glass-tube G_3. $d = 8.640$ mm.

With Mercury. $l_1 = 1450$ mm. $l = 548$ mm.

q	τ	$h_1 - h_2$	t	v	h	$x = \frac{vd}{\nu}$	$y = \frac{h}{\left(\frac{4v^2}{gd}\right)}$
kg	sec.	cm Hg	°C	cm/sec.	cm Hg/cm		
6	61.4	0.17					
6	61.4	0.18					
6	61.4	0.175	20.2	12.30	0.00319	9080	0.00448
7	48.8	0.32					
7	49.1	0.32					
7	48.95	0.32	20.0	18.00	0.00584	13290	382
10	54.4	0.49					
10	54.4	0.47					
10	54.4	0.48	19.8	23.14	0.00876	17090	347
15	61.2	0.88					
15	61.6	0.87					
15	61.4	0.875	19.6	30.76	0.0160	22720	357
20	64.0	1.29					
20	64.0	1.26					
20	64.0	1.275	20.0	39.34	0.0233	29050	319

q	τ	$h_1 - h_2$	t	v	h	$x = \frac{vd}{\nu}$	$y = \frac{h}{\left(\frac{4v^2}{gd}\right)}$
kg	sec.	cm Hg	°C	cm/sec.	cm Hg/cm		
20	50.6	1.99					
20	50.5	1.97					
20	50.55	1.98	19.9	49.82	0.0361	36790	309
30	57.2	3.20					
30	57.2	3.20					
30	57.2	3.20	19.8	66.03	0.0584	48760	284
40	64.9	4.25					
40	64.6	4.32					
40	64.75	4.285	19.7	77.78	0.0782	57440	274
40	52.9	6.40					
40	53.0	6.12					
40	52.95	6.26	19.9	95.10	0.114	70230	268
40	46.2	8.06					
40	46.0	8.10					
40	46.1	8.08	20.0	109.2	0.148	80670	262
40	38.4	12.3					
40	39.4	11.2					
40	38.9	11.75	20.1	129.6	0.214	95670	271

With Water. $l_1 = 901$ mm. $l = 548$ mm.

kg	sec.	cm H_2O	°C	cm/sec.	cm H_2O/cm		
5	240.7	1.81					
		1.83					
5	240.7	1.82	17.2	35.42	0.0332	2840	0.00560
6	196.5	3.60					
		3.72					
6	196.5	3.66	17.2	52.15	0.0668	4170	521
8	186.6	6.50					
		6.50					
8	186.6	6.50	17.2	73.12	0.119	5850	470
10	170.8	11.7					
		11.6					
10	170.8	11.65	17.2	99.83	0.213	7990	454
15	195.3	17.4					
		18.0					
15	195.3	17.7	17.2	130.9	0.323	10470	400

q	τ	$h_1 - h_2$	t	v	h	$x = \frac{vd}{\nu}$	$y = \frac{h}{\left(\frac{4v^2}{gd}\right)}$
kg	sec.	cm H_2O	°C	cm/sec.	cm H_2O/cm		
15	159.1	25.7					
15	158.9	25.7					
15	159.0	25.7	17.3	160.9	0.469	12870	384
20	166.3	39.2					
20	166.1	39.2					
20	166.2	39.2	17.4	205.3	0.715	16430	360
25	159.1	62.3					
25	158.9	62.5					
25	159.0	62.4	17.4	268.1	1.139	21640	336
30	151.5	93.5					
30	151.3	93.5					
30	151.4	93.5	17.3	337.5	1.708	27020	318
40	156.7	145.0					
40	156.5	145.0					
40	156.6	145.0	17.2	435.8	2.646	34830	295
50	167.7	191.0					
50	167.7	191.0					
50	167.7	191.0	17.3	508.4	3.483	40660	286

Appendix B.

Steel-tube S_1. $d = 15.05$ mm. $l_1 = 1500$ mm. $l = 1000$ mm.

With Mercury. Direction of flow 1—2.

q	τ	$h_1 - h_2$	t	v	h	$x = \frac{vd}{\nu}$	$y = \frac{h}{\left(\frac{4v^2}{gd}\right)}$
kg	sec.	cm Hg	°C	cm/sec.	cm Hg/cm		
70	82.0	0.80					
70	82.8	0.88					
70	82.4	0.84	19.3	35.23	0.0087	45340	0.00259
60	70.2	0.85					
60	70.4	0.75					
60	70.3	0.80	19.8	35.39	0.0083	45550	245
60	58.8	1.20					
60	59.2	1.30					
60	59.0	1.25	19.8	42.17	0.0128	54270	266

q	τ	$h_1 - h_2$	t	v	h	$x = \frac{vd}{\nu}$	$y = \frac{h}{\left(\frac{4v^2}{gd}\right)}$
kg	sec.	cm Hg	°C	cm/sec.	cm Hg/cm		
50	49.2	1.07					
50	49.0	1.19					
50	49.1	1.13	18.8	42.22	0.0116	54340	0.00241
60	43.3	2.25					
60	43.3	2.20					
60	43.3	2.23	19.7	57.48	0.0226	73980	253
60	42.8	2.08					
60	43.1	2.10					
60	42.95	2.09	18.9	57.93	0.0212	74560	233
60	35.6	3.20					
60	35.3	3.30					
60	35.45	3.25	19.5	70.21	0.0328	90360	246
70	40.6	3.20					
70	41.0						
70	40.8	3.20	19.1	71.16	0.0323	91580	236
60	26.4	5.4					
60	26.9	5.2					
60	26.65	5.3	19.5	93.39	0.0533	120200	226
70	30.8	5.3					
70	30.7						
70	30.75	5.3	19.6	94.43	0.0533	121500	221
60	21.8	7.2					
60	21.7						
60	21.75	7.2	19.4	114.4	0.0723	147300	204

With Water. Direction of flow 1—2.

kg	sec.	cm H_2O	°C	cm/sec.	cm H_2O/cm		
25	171.2	7.0					
25	171.2	7.0	16.9	82.04	0.070	11330	0.00384
25	109.6	15.3					
25	109.6	15.3	17.0	128.1	0.153	17700	345
50	161.4	26.0					
50	161.2	25.8					
50	161.3	25.9	17.0	174.5	0.259	24100	314

q	τ	$h_1 - h_2$	t	v	h	$x = \frac{vd}{\nu}$	$y = \frac{h}{\left(\frac{4v^2}{gd}\right)}$
kg	sec.	cm H_2O	°C	cm/sec.	cm H_2O/cm		
75	196.5	36.9					
75	196.1	36.7					
75	196.3	36.8	17.2	214.8	0.368	29960	0.00295
75	168.5	48.1					
75	168.7	48.1					
75	168.6	48.1	17.3	250.0	0.481	34800	284
75	148.5	60.3					
75	147.9	60.1					
75	148.2	60.2	17.5	284.5	0.602	40020	274
100	170.4	78.5					
100	170.2	78.5					
100	170.3	78.5	17.7	329.8	0.785	46400	266
100	150.7	96.7					
100	150.9	96.5					
100	150.8	96.6	17.8	372.7	0.966	52420	257
100	140.9	109.4					
100	141.1	109.4					
100	141.0	109.4	17.8	398.7	1.094	56080	254

With Water. Direction of flow 2—1.

kg	sec.	cm H_2O	°C	cm/sec.	cm H_2O/cm		
25	255.5	3.61					
25	255.7	3.65					
25	255.6	3.63	16.3	55.34	0.0363	7500	0.00437
50	277.3	10.21					
50	277.5	10.23					
50	277.4	10.22	16.4	101.3	0.1022	13860	367
50	249.6	12.00					
50	248.8	12.00					
50	249.2	12.00	16.4	112.6	0.1200	15420	350
50	212.0	16.1					
50	212.2	15.9					
50	212.1	16.0	16.4	132.5	0.160	18130	337
50	180.0	21.6					
50	179.6	21.4					
50	179.8	21.5	16.4	156.4	0.215	21400	325

q	τ	$h_1 - h_2$	t	v	h	$x = \frac{vd}{\nu}$	$y = \frac{h}{\left(\frac{4v^2}{gd}\right)}$
kg	sec.	cm H_2O	°C	cm/sec.	cm H_2O/cm		
50	143.9	31.3					
50	144.1	31.7					
50	144.0	31.5	16.6	195.3	0.315	26720	0.00305
75	189.6	39.4					
75	189.8	39.6					
75	189.7	39.5	16.8	222.2	0.395	30700	295
75	158.3	54.4					
75	157.9	54.2					
75	158.1	54.3	17.0	266.8	0.543	36800	282
100	180.0	71.9					
100	180.0	71.7					
100	180.0	71.8	16.9	312.8	0.718	43180	271
100	158.1	90.3					
100	158.3	90.5					
100	158.2	90.4	16.9	355.2	0.904	49040	264
100	138.1	114.3					
100	138.1	114.3					
100	138.1	114.3	16.9	407.0	1.143	56200	254

Appendix C.

The Bernoulli Correction.

The glass-tube G_1 gave, with a flow in the one direction, a curve deviating markedly from the one which was expected. The cause was sought in variations in the cross-section of the tube. With stream-line motion of the fluid a difference of the diameters at the two ends of the research-part of the tube would, according to Bernoulli's theorem, give rise to a pressure-difference. And from many experiences it may be concluded that the same pressure-difference comes into existence with a turbulent flow. It is therefore easy to derive an expression for the correction Δy, the Bernoulli correction, which the said difference of diameters will give rise to. With a horizontal tube, according to Bernoulli's theorem

$$1^\circ \qquad p + \frac{1}{2}\varrho v^2 = \text{constant}.$$

If the volume Q is streaming through the tube per sec.

2°
$$v = \frac{Q}{\frac{\pi}{4} d^2}.$$

From 1° and 2° follows that an increase of the diameter Δd in the direction of the flow will cause a pressure difference Δp expressed by

3°
$$\Delta p = \varrho v^2 \cdot \frac{2 \Delta d}{d}.$$

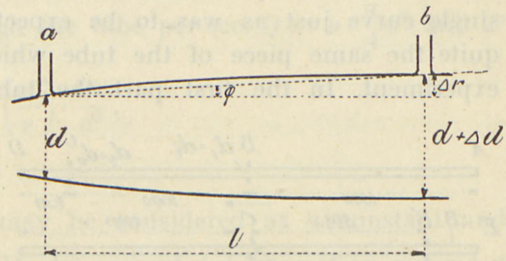


Fig. 15.

Thus the observed pressure-drop is too small by an amount $\Delta h = \frac{\Delta p}{\varrho g}$. Or to the pressure gradient h the correction $\Delta h = \frac{\Delta h}{l} = \frac{2 v^2}{g d} \cdot \frac{\Delta d}{l}$ should be added. The corresponding correction Δy of the ordinates in the y—x-curves above is given by

4°
$$\Delta y = \frac{\Delta h \cdot g d}{4 v^2} = \frac{\Delta d}{2 l} = \frac{\Delta r}{l}$$

r indicating the radius of the tube. The correction to be applied to the dimensionless quantity y is thus simply the increase of radius per unit-length of the tube viz.

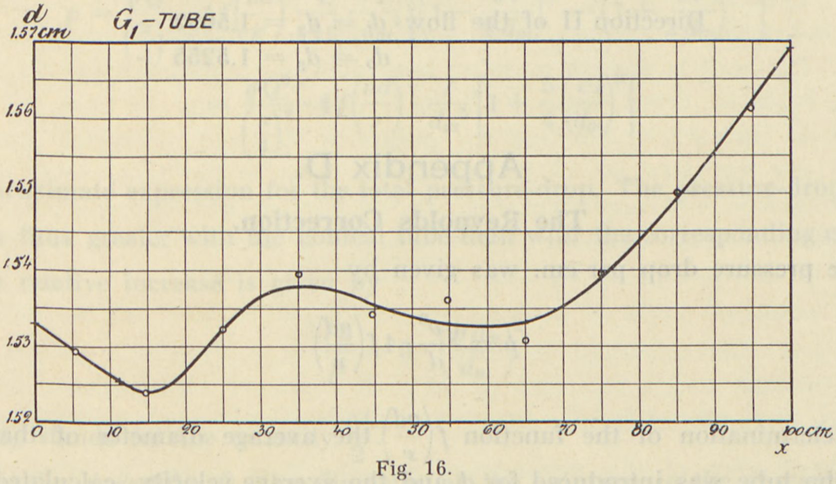


Fig. 16.

the angle φ of fig. 15. The correction is independent of the profile of the tube between the two manometer-bores. It has the same absolute value for all abscissae and is identical for water and mercury. Thus, if we only want to verify the law of similarity, the correction need not be drawn into account. If, on the contrary, a correct determination of the function $y = f\left(\frac{vd}{\nu}\right)$ is aimed at, the correction must be applied.

The theory of the correction was verified by means of the tube G_1, the caliber-curve of which is reproduced in fig. 16. Experiments were carried out with the tube for flow in both directions. The results are represented in fig. 12. If the points determined by the original observations are corrected, two new series of points are obtained which, apart from an anomaly in one of the two series, determine one single curve just as was to be expected. It should be remarked that it was not quite the same piece of the tube which was used in the two parts of the twin-experiment. In the first part the tube had the aspect shown in fig. 17 *a*. The research-length was *BC*. After the experiment the tube burst at *B*. A fresh hole was drilled at *B'* fig. 17 *b*. The piece *AB*, fig. 17 *a* was blown together with the other end of the tube at *D* as *DA* fig. 17 *b* and the tube was furthermore prolonged with the piece *EB*.

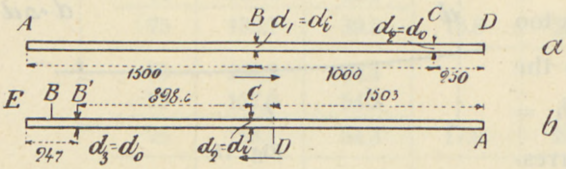


Fig. 17.

The experiment with the opposite direction of flow was then carried out. Owing to the alterations the Bernoulli corrections were thus a little different in the two parts of the experiment. They were calculated from the following values of the diameters:

Direction I of the flow	$d_i = d_1 = 1.5330$ cm.
	$d_o = d_2 = 1.5692$ -
Direction II of the flow	$d_i = d_2 = 1.5692$ -
	$d_o = d_3 = 1.5255$ -

Appendix D.

The Reynolds Correction.

The pressure drop per cm. was given by

1° $$p = \frac{\varrho v^2}{d} \cdot 4 f\left(\frac{vd}{\nu}\right).$$

In the determination of the function $f\left(\frac{vd}{\nu}\right)$ the average diameter of the research-part of the tube was introduced for d and the average velocity, calculated from the average diameter and from the quantity of liquid emitted per sec., was entered for v. If the tube is cylindrical the correct value of $f\left(\frac{vd}{\nu}\right)$ is obtained, but if variations occur in the diameter a false value is in most cases derived because p, apart from the Bernoulli correction, will, with a tube of varying diameter, assume a value different from that observed with one of a constant diameter equal to the average diameter of the irregular tube. In the following an expression is derived for the correction which may perhaps be termed the Reynolds correction. The pressure-drop ∂p in the length dx of the irregular tube may presumably be expressed by

$$2^\circ \qquad \partial p = \frac{\varrho v^2}{d} \cdot 4 f\left(\frac{vd}{\nu}\right) \cdot \partial x.$$

If a volume Q of the fluid is passing through the tube per sec. $Q = v \cdot \frac{\pi}{4} d^2$ and 2° may be written:

$$2^{\circ\prime} \qquad \partial p = \frac{\varrho Q^2}{\left(\frac{\pi}{4}\right)^2} \cdot 4 f\left(\frac{vd}{\nu}\right) \cdot \frac{\partial x}{d^5}.$$

With small variations of the caliber $f\left(\frac{vd}{\nu}\right)$ may be considered as a constant and the pressure-loss for the whole tube may be determined by integration, provided the variation of d along the tube is known. We confine ourselves to the case of a conical tube. In this case we may write

$$3^\circ \qquad d = d_m + c\,x.$$

If d_m denotes the diameter at the middle-point of the tube of length l, x varies from $-\frac{l}{2}$ to $+\frac{l}{2}$. Introducing the expression 3° for d in $2^{\circ\prime}$ and integrating we get

$$4^\circ \qquad \begin{aligned} p &= \frac{\varrho Q^2}{\left(\frac{\pi}{4}\right)^2} \cdot 4 f\left(\frac{vd}{\nu}\right) \cdot \frac{1}{4c} \cdot \frac{1}{d_m^4}\left[\left(1 - \frac{c\,l}{2\,d_m}\right)^{-4} - \left(1 + \frac{c\,l}{2\,d_m}\right)^{-4}\right] \\ &= \frac{\varrho Q^2}{\left(\frac{\pi}{4}\right)^2} \cdot 4 f\left(\frac{vd}{\nu}\right) \cdot \frac{l}{d_m^5}\left[1 + \frac{5}{4}\left(\frac{c\,l}{d_m}\right)^2\right] \end{aligned}$$

as an approximate expression for the total pressure-drop. The pressure-drop per cm., $\dot{p} = \frac{p}{l}$, is thus greater with the conical tube than with the corresponding cylindrical tube. The relative increase is given by

$$5^\circ \qquad \frac{\varDelta \dot{p}}{\dot{p}} = 5 \cdot \left(\frac{\varDelta d_m}{d_m}\right)^2$$

where $\varDelta d_m$ stands for the quantity $\frac{c\,l}{2}$.

With the glass-tube G_1 the diameter at the entrance to the research part of the tube was in the first part of the experiment 1.5330 cm., and at the end of the same 1.5692 cm. The tube was not exactly conical, the average-diameter being 1.538 and the mean-value of the inlet- and outlet-diameter 1.551. If, however, the tube were conical we would have $\varDelta d_m = 0.018$ cm., $\frac{\varDelta d_m}{d_m} = 0.012$ and $5\left(\frac{\varDelta d_m}{d_m}\right)^2 = 0.72 \cdot 10^{-3}$. The Reynolds correction is thus very small even with this rather irregular tube and it may presumably in most cases be disregarded.

Appendix E.

A Determination of the Roughness of the Glas-Tubes.

An examination of the experiments carried out with the glass-tubes shows that the curves, representing the said experiments, lie the higher the less the diameter of the tube is. With a view to orientation values of $y = f\left(\frac{vd}{\nu}\right)$ corresponding to $\frac{vd}{\nu}$ equal to 10000, 30000 and 50000 have been collected in the following table. The y-values exhibit, as will be seen, a marked rise with decreasing diameter.

	d	$y \cdot 10^5$			α	β
		10000	30000	50000		
G_1	1.538	416	305	269	0.00153	0.240
G_2	1.360	405	303	271	0.00170	0.210
G_3	0.8640	408	309	280	0.00180	0.205
G_4	0.5653	425	325	295	0.00195	0.210
G_5	0.3372	455	350	316	0.00205	0.235

In the tabel the values α and β of the research-curve are introduced. While β does not show any pronounced variation, α increases when the diameter decreases. The explanation of the rise in y and α is to be sought in the fact of the friction, and thus the pressure-loss, depending on the relative roughness of the tube-wall. The wall is always more or less rough. As a measure for the roughness the mean hight or mean depth s of the irregularities may be used. The relative roughness in then s/r, r denoting the radius of the tube. Provided the absolute roughness s is the same for all the glass-tubes used in the experiments above, the relative roughness will vary from one tube to another and this will cause the friction to vary. It seems that the relative roughness can be drawn into account by writing α in the formula 1° pg. 392 (10) under the form

$$\alpha = 0.0012 + \frac{1}{2}\sqrt{\frac{2k}{d}} \qquad 1^\circ$$

where k is proportional to s[1]). According to this expression α varies linearly with $\sqrt{\frac{1}{d}}$. A verification of this dependency is undertaken in fig. 18. We will assume the straight line drawn to represent the dependency and find for k the value $0.57 \cdot 10^{-6}$ cm. On the basis of this value the α of the extrapolated water-curve of the G_6-experiment was calculated. For β the mean value derived from the experiments with G_1 to G_5 was used. The k-value lies inside the limits $0.2—0.8 \cdot 10^{-5}$ which are generally assumed for the roughness of glass. It should however be

[1]) Compare: R. V. Mises. Elemente der technischen Hydromechanik Teil I pg. 61—62.

noted that an exact test on the theory of roughness demands a calibration of all the tubes, while here a calibration was only carried out with G_1 and G_3. The Bernoulli correction assumes a considerable size, with G_1. With G_3 it was only about 1 % of y. The regularity of the manner in which y and α vary with d is relatively high and presumably justifies us in assuming that the Bernoulli correction is also small for the remaining tubes G_3, G_4 and G_5, thus making it allowable to use the experiments for the determination of the roughness.

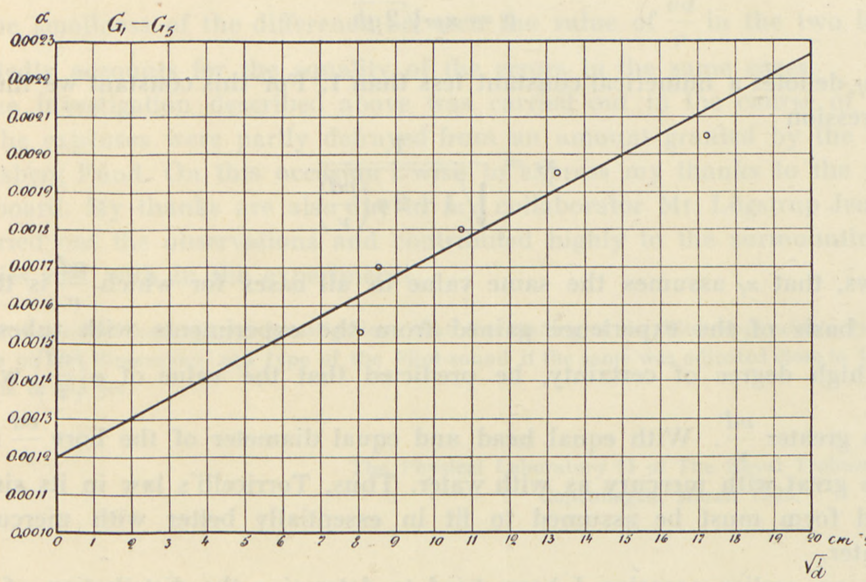


Fig. 18.

Appendix F.

Note on the Law of Similarity as applied to other Systems.

The law of similarity applies to all kinds of hydrodynamic systems. The fundamental condition for its application is, as indicated above, that the systems are geometrically similar. The law must therefore apply, for instance, to similar nozzles.

If the pressure-drop expressed as a liquid column is denoted by Δh we must have

$$1^{\circ} \qquad \Delta h = \frac{v^2}{2g} \cdot \varphi\left(\frac{vd}{\nu}\right).$$

According to Torricelli's law v is determined by

$$2^{\circ} \qquad v = \sqrt{2g(h - \Delta h)}.$$

If the expression for Δh is introduced in 2° we get

3° $$v = \frac{1}{\sqrt{1 + 2\varphi\left(\frac{vd}{\nu}\right)}} \cdot \sqrt{2gh}.$$

Generally the friction in the jet-hole is drawn into account by writing

4° $$v = \varkappa_f \cdot \sqrt{2gh}$$

where $\varkappa_f$ denotes a numerical constant less than 1. For this constant we thus derive the expression

5° $$\varkappa_f = \frac{1}{\sqrt{1 + 2\varphi\left(\frac{vd}{\nu}\right)}}.$$

It follows, that $\varkappa_f$ assumes the same value in all cases for which $\frac{vd}{\nu}$ is the same. On the basis of the experience gained from the experiments with tubes it can, with a high degree of certainty, be predicted that the value of $\varphi\left(\frac{vd}{\nu}\right)$ will prove less, the greater $\frac{vd}{\nu}$. With equal head and equal diameter of the bore $\frac{vd}{\nu}$ is ab. 10 times as great with mercury as with water. Thus, Torricelli's law in its simple uncorrected form must be assumed to fit in essentially better with mercury than with water.

On an earlier occasion I have tried to determine the distribution of velocity in liquid jets by means of a Pitot-tube. We will think of the Pitot-tube as applied to geometrically similar systems and will express the reading p of the tube as

$$p = \frac{1}{2}\varrho v^2\left(1 - \psi\left(\frac{vd}{\nu}\right)\right).$$

If $\frac{vd}{\nu}$ increases the ψ-function will, in all probability, approach asymptotically to zero. The greater the value of $\frac{vd}{\nu}$ the less the error committed in calculating v from the expression $p = \frac{1}{2}\varrho v^2$. With the same velocity, jet-diameter and Pitot-sound the error will thus prove essentially less with a mercury-jet than with a water-jet. This was confirmed by the investigation alluded to. With a water-jet of 2 mm. and with a velocity of ab. 360 cm./sec., the said velocity was found to be 25 % too low just outside the jet-hole, while with a mercury-jet of the same thickness and a velocity of 200 cm./sec. the error was only 1—2 %. Finally it was found that the error was also ab. 1 % with a water jet of 2 mm. and with a velocity of ab. 2000 cm./sec. The Pitot-sound was in all three cases a glass-tube drawn to a fine point with an aperture of 0.5—0.7 mm.[1]). If $\frac{vd}{\nu}$ is calculated in the three cases, d being taken as the diameter of the jet and the temperature being judged at 20°, we get

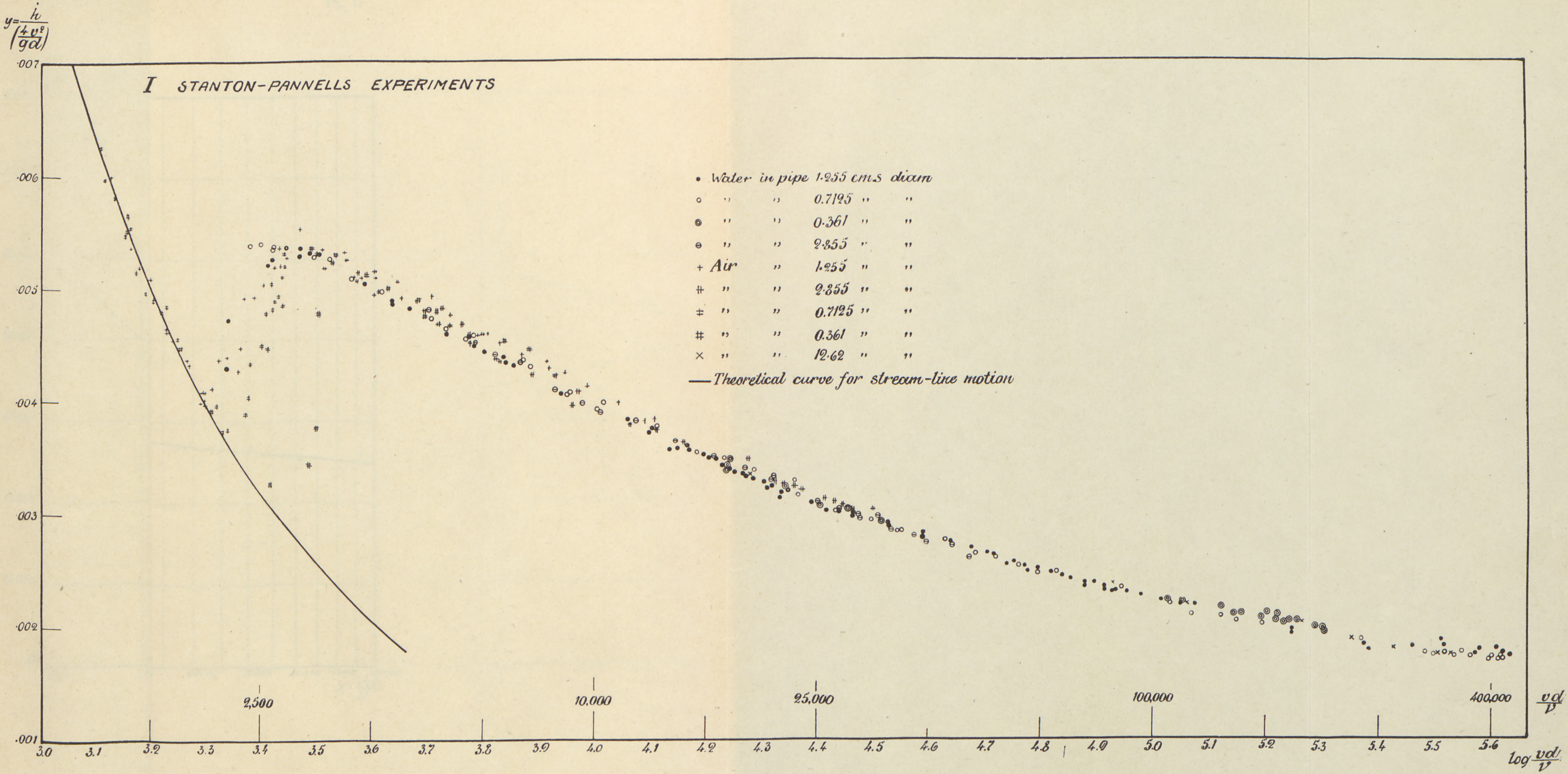
Water-jet	360 cm./sec.	$\frac{vd}{\nu}$	= 7200
Mercury-jet	200 —	—	= 34000
Water-jet	2000 —	—	= 40000

The smallness of the difference between the value of $\frac{vd}{\nu}$ in the two last cases undoubtedly accounts for the equality of the errors in the same cases.

The investigation described above was carried out in the course of the year 1924. The expenses were partly defrayed from an amount granted by the board of the Carlsberg Fund. On this occasion I wish to express my thanks to the members of the board. My thanks are also due to my collaborator Mr. Lögstrup Jensen who has carried out the observations and contributed highly to the surmounting of the difficulties met with in the experiments.

[1]) It appeared from special investigations that the reading of the Pitot-apparatus only depended very little on the dimensions and type of the Pitot-sound if the same was adjusted close to the jet-hole in the axis of the jet.

The Physical Laboratory II of The Royal Technical College.
Copenhagen, March 1925.


I STANTON-PANNELLS EXPERIMENTS
• Water in pipe 1.255 cms diam
○ " " 0.7125 " "
⊙ " " 0.361 " "
⊖ " " 2.855 " "
+ Air " 1.255 " "
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" " 0.361 " "
× " " 12.62 " "
— Theoretical curve for stream-line motion
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·006
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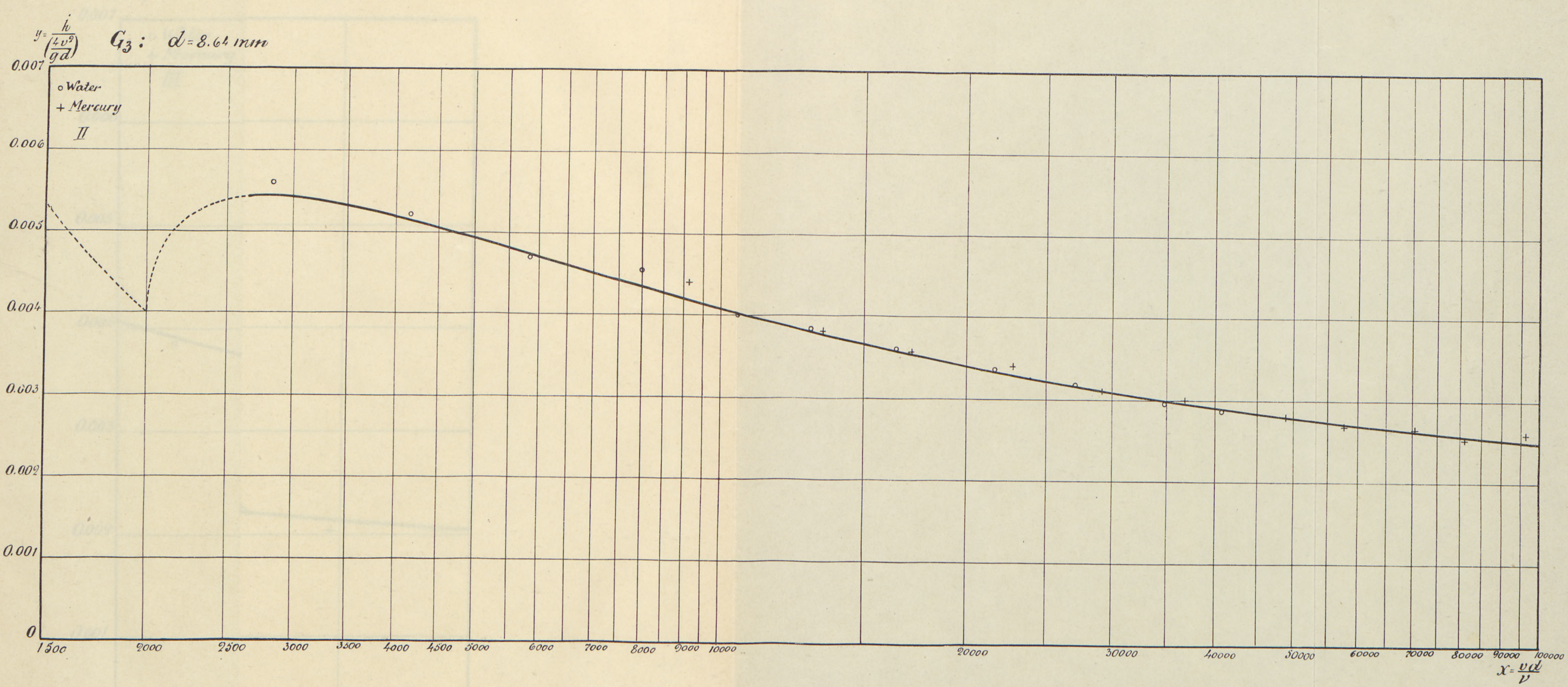
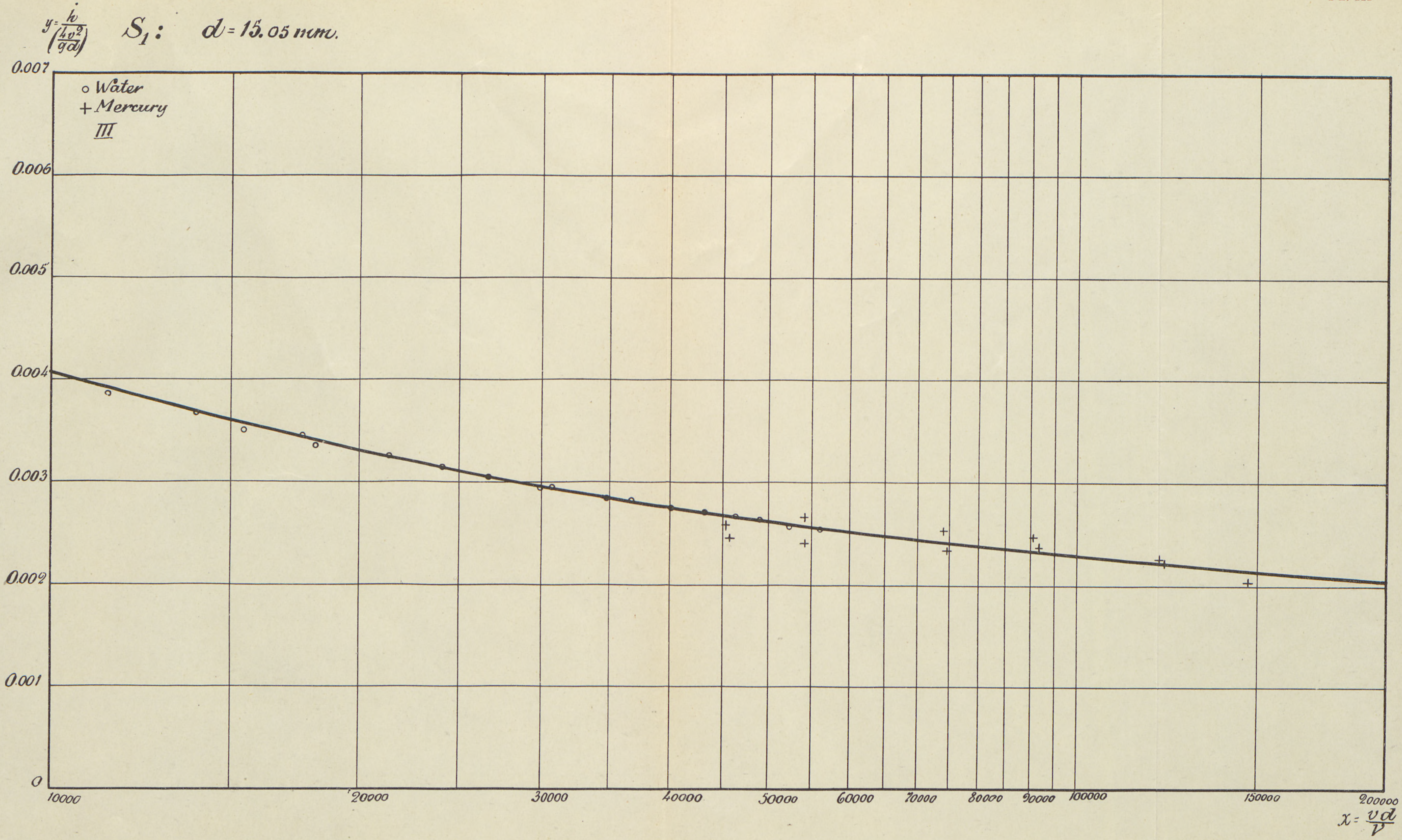
5.5
5.6


$y = \frac{h}{\left(\frac{4v^2}{gd}\right)}$
G_3 : $d = 8.64$ mm
o Water
+ Mercury
II
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
1500
2000
2500
3000
3500
4000
4500
5000
6000
7000
8000
9000
10000
20000
30000
40000
50000
60000
70000
80000
90000
100000
$x = \frac{vd}{\nu}$



$y = \frac{h}{\left(\frac{4v^2}{gd}\right)}$
S_1: $d = 15.05$ mm.
○ Water
+ Mercury
III
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
10000
20000
30000
40000
50000
60000
70000
80000
90000
100000
150000
200000
$x = \frac{vd}{\nu}$